MATHEMATICAL ANALYSIS OF GENERALIZED BASIC HYPERGEOMETRIC FUNCTION IN QUANTUM CALCULUS

Synopsis of the
Research Proposal
Submitted for the Registration in the

Ph.D. Program
(Part-Time) in
Mathematics

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1. Introduction

1.1 q-analogue of special functions: The q-analog of a special function is a mathematical concept that generalizes a classical special function by introducing a deformation parameter q. These q-analogs arise in the context of q-calculus or quantum calculus, which is a non-commutative version of calculus.

1.2. q-analogue of Gamma and Beta function: First by Thomae (1869) and then by Jackson (1905b) introduced the q-gamma function given by

$$\Gamma_q(\alpha) = \frac{(q;q)_\infty}{(q^\alpha;q)_\infty}(1-q)^{1-\alpha} = \frac{(\alpha;q)_{\alpha-1}}{(1-q)^{\alpha-1}},\ (\alpha \in \mathbb{C}, |q| < 1)$$

Here we take the principal value of q^α and$(1-q)^{1-\alpha}$. Then $\Gamma_q(\alpha)$ is a meromorphic function with poles at$\alpha = -n, n \in \mathbb{N}_0$. Since $\Gamma_q(\alpha)$ has no zeros, then $\frac{1}{\Gamma_q(\alpha)}$ is an entire function with zeros at $\alpha = -n, n \in \mathbb{N}_0$. Obviously,

$$\Gamma_q(n) = \frac{(q;q)_{n-1}}{(1-q)^{n-1}},\ n \in \mathbb{N}$$

It is known that for $x > 0, \Gamma_q(x)$ is the unique logarithmically convex function that satisfies the functional equation

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x),\ \Gamma_q(1) = 1$$

The q-beta function is defined by

$$\mathfrak{B}_q(x,y) = \int_0^1 z^{x-1}(qz;q)_{y-1}d_qz = \int_0^1 z^{x-1}\frac{(zq;q)_\infty}{(zq^y;q)\infty}d_qz,\ (\Re(x) > 0, \Re(y) > 0).$$

Askey (1978) proved

$$\mathfrak{B}_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)},\ (\Re(x) > 0, \Re(y) > 0).$$

1.3 q-analogue of Gauss hypergeometric function:

The $(q-)$ analogue of the well-known Gaussian hypergeometric series is given by:

$$1+\frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}z+\frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}z^2+\cdots,$$

where $|q|<1$ and $|z|<1$. We can rewrite (20) as:

$$ {}_2\Phi_1\left[\begin{matrix}q^a, q^b;\\ q^c;\end{matrix}\quad q,z\right]\equiv {}_2\Phi_1\left[\begin{matrix}a,b;\\ c;\end{matrix}\right] \text{ or } {}_2\Phi_1[a,b;c;q,z]$$

The development of the theory of basic hypergeometric functions of one and more variables rose from the generalization of the ordinary Gaussian hypergeometric series ${}_2\,\mathrm{F}_1(\cdot)$ to the basic $(q-)$ Gauss hypergeometric series ${}_2\Phi_1(\cdot)$.

Both the basic (or q-) hypergeometric function theory and the ordinary hypergeometric function theory are in-depth. Numerous areas of the mathematical, physical, and engineering sciences make a great deal of use of the ideas of the basic hypergeometric functions of one or more variables. Number theory, partition theory, the theory of elliptic functions, combinatorial analysis, Lie theory, fractional calculus, integral transforms, high-energy particle physics, quantum theory, Moak-theta functions of various orders, statistics, etc. are a few of the fields where the basic hypergeometric functions of one and more variables have found extensive applications.

Heine (1878) defined the basic analogue of the Gaussian hypergeometric function in the following way, using the notion of the basic number:

$$ {}_2\Phi_1\left[\begin{matrix}a,b;\\ q,z\\ c;\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(q;q)_n(c;q)_n}z^n,$$

Where $|q|<1, |z|<1$ and for real or complex ' a ', the q-shifted factorial factorial $(a;q)_n$ is defined.

1.4 The Generalized Basic Hypergeometric Function:

The generalized basic hypergeometric function ${}_r\Phi_s(\cdot)$ is a generalization of the series ${}_2\Phi_1(\cdot)$. It is denoted and defined cf. Gasper and Rahman (1990), as:

$$ {}_r\Phi_s\left[\begin{matrix}a_1,\ldots,a_r;\\ b_1,\ldots,b_s;\end{matrix}\quad q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n\,z^n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{1+s-r}, $$

where $\binom{n}{2}=\frac{n(n-1)}{2}$ and $|q|<1$.

The series ${}_r\Phi_s(\cdot)$ converges absolutely for all ' z ' if $r\le s$ and for $|z|<1$ if $r=s+1$. The series also converges absolutely if $|q|>1$ and

$$ |z|<\frac{|b_1b_2..b_s|}{|a_1a_2..a_r|} $$ Another form of the generalized basic hypergeometric series ${}_r\Phi_s(\cdot)$ cf. Slater (1966) is given by:

$$ {}_r\Phi_s\left[\begin{matrix}a_1,\ldots,a_r;\\ b_1,\ldots,b_s;\end{matrix}\quad q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n\,z^n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}, $$

where $|z|<1$, $|q|<1$.

1.5 q-analogue of generalized H-function: In terms of the basic contour integral of the Mellin Barnes type, Saxena and Kumar (1995) developed a basic analogue of the I-function in the following manner:

$$ I_{A_i,B_i}^{m,n}\left[x;q\;\middle|\;\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,A_i}\\ (b_j,\beta_j)_{1,m},(b_j,\beta_j)_{m+1,B_i}\end{matrix}\right] $$
$$ =\frac{1}{2\pi\omega}\int_C\frac{\prod_{j=1}^{m}G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n}G(q^{1-a_j+\alpha_j s})\pi x^s}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{B_i}G\left(q^{1-b_{ji}+\beta_{ji}s}\right)\prod_{j=n+1}^{A_i}G(q^{a_{ji}-\alpha_{ji}s})\right\}G(q^{1-s})\sin\pi s}ds, $$

where $0\le m\le B_i;\ 0\le n\le A_i;\ i=1,2,\cdots,r;\ r$ is finite; $\omega=\sqrt{-1}$; and

$$ G(q^a)=\left\{\prod_{n=0}^{\infty}(1-q^{a+n})\right\}^{-1}=\frac{1}{(q^a;q)_\infty}, $$
$$ (a;q)_n=\frac{G(q^{a+n})}{G(q^a)}=\frac{(a;q)_\infty}{(aq^n;q)_\infty}. $$

Also $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive and a_j, b_j, a_{ji}, b_{ji} are complex numbers.

The contour of integration c runs from $-i\infty$ to $+i\infty$ in such a manner that all the poles of $G\left(q^{b_j-\beta_j s}\right); 1 \leq j \leq m$, are to its right, and those of $G(q^{1-a_j+\alpha_j s}), 1 \leq j \leq n$, are to its left and at least some $\varepsilon > 0$ distance away from the contour c. The basic integral converges if $\mathrm{Re}[s\log(x) - \log\sin\pi s] < 0$, for large values of $|s|$ on the contour, that is if $|\arg x| < \pi$. It may be observe that the contour of integration c can be replaced by other suitably indented contours parallel to the imaginary axis.

It is interesting to observe that for $r = 1, A_1 = A; B_1 = B$; definition (45) yields the q-analogue of the Fox's H-function due to Saxena et al. (1983), namely

$$H_{A,B}^{m,n}\left[x; q \,\middle|\, \begin{matrix}(a,\alpha)\\(b,\beta)\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-\beta_j s}\right)\prod_{j=1}^{n} G(q^{1-a_j+\alpha_j s})\pi x^s}{\prod_{j=m+1}^{B} G\left(q^{1-b_j+\beta_j s}\right)\prod_{j=n+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\,ds.$$

If we set $\alpha_i = \beta_j = 1, \forall i$ and j in the above equation , then it reduces to a basic analogue of Meijer's G-function defined by Saxena et al. (1983), namely

$$H_{A,B}^{m,n}\left[x; q \,\middle|\, \begin{matrix}(a,1)\\(b,1)\end{matrix}\right] \equiv G_{A,B}^{m,n}\left[x; q \,\middle|\, \begin{matrix}a_1,\cdots,a_A\\b_1,\cdots,b_B\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_C \frac{\prod_{j=1}^{m} G\left(q^{b_j-s}\right)\prod_{j=1}^{n} G(q^{1-a_j+s})\pi x^s}{\prod_{j=m+1}^{B} G\left(q^{1-b_j+s}\right)\prod_{j=n+1}^{A} G(q^{a_j-s})G(q^{1-s})\sin\pi s}\,ds,$$

Where $0 \leq m \leq B, 0 \leq n \leq A$ and $\mathrm{Re}[s\log(x) - \log\sin\pi s] < 0$.

${}_2\Phi_1(\cdot)$. Gasper (1990) denoted and defined as:

$${}_r\Phi_s\left[\begin{matrix}a_1,\ldots,a_r;\\ b_1,\ldots,b_s;\end{matrix} q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1;q)_n\cdots(a_r;q)_n z^n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{1+s-r},$$

where $\binom{n}{2} = \frac{n(n-1)}{2}$ and $|q| < 1$.

The series ${}_r\Phi_s(\cdot)$ converges absolutely for all ' z ' if $r \leq s$ and for $|z| < 1$ if $r = s + 1$. The series also converges absolutely if $|q| > 1$ and

$$|z| < \frac{|b_1 b_2 \dots b_s|}{|a_1 a_2 \dots a_r|}$$

Alternative form of the generalized basic hypergeometric series ${}_r\Phi_s(\cdot)$ cf. Slater (1966) is given by:

$${}_r\Phi_s\left[\begin{matrix} a_1, \dots, a_r; \\ b_1, \dots, b_s; \end{matrix} q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n z^n}{(q; q)_n (b_1; q)_n \cdots (b_s; q)_n}.$$

1.6 q-analogue of Mittag-leffler function: The q-extension of ordinary calculus is known as the q-calculus. Recently, the theory of q-calculus operators has been used in the areas of ordinary fractional calculus, optimal control problems, solving the q difference and q-integral equations, and q-transform analysis. According to Gasper and Rahman (2004), the usual exponential functions are defined in q-analogue form in the theory of q-calculus.

$$e_q(x) = \frac{1}{(x; q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} \ (|x| < 1),$$

$$E_q(x) = (-x; q)_\infty = \sum_{n=0}^{\infty} \frac{q^{(2)}}{(q; q)_n} x^n.$$

Or, applying the q-form of the Taylor theorem, by

$$e_q(x) = e_q(c) \sum_{n=0}^{\infty} \frac{x^n (c/x; q)_n}{(q; q)_n} \ (c \in \mathbb{R}, |c| < |x|),$$

$$E_q(x) = E_q(c) \sum_{n=0}^{\infty} \frac{q^{(2)}}{(-c; q)_n} \frac{x^n (c/x; q)_n}{(q; q)_n}, \ (c, x \in \mathbb{R}).$$

In 2007, Rajkovic et al. (2007) introduced the small q-Mittag-Leffler function and the big q-Mittag-Leffler function by

$$e_{q;\alpha,\beta}(x;c)=\sum_{n=0}^{\infty}\frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}},\ (|c|<|x|),$$

$$(q,x,c,\alpha,\beta\in\mathbb{C};\Re(\alpha),\Re(\beta)>0,|q|<1),$$

$$\binom{\alpha n+\beta-1}{2}_{x_{q;\alpha,\beta}(x;c)}=\sum_{n=0}^{\infty}\frac{q}{(-c;q)_{\alpha n+\beta-1}}\frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}},$$

Purohit and Kalla (2011) introduced q-analogue of the generalized Mittag-Leffler.

1.7 The q-integral and q-derivative operators:

The q-fractional calculus is the q-extension of the ordinary fractional calculus. The theory of q-fractional integral and differential operators in recent past has become one of the fast immerging field, having wide scope of applications in the areas like ordinary fractional calculus, Combinatorial analysis, solution of the q-difference and q-integral equations, and q-transform analysis etc.

The q-derivative operator is defined as:

$$D_q f(x)=\frac{f(x)-f(xq)}{(1-q)x},$$

for fixed q with $0<|q|<1$.

In the year 1951, Jackson (1951) defined q-integrals in the following manner:

$$\int_0^x f(t)\,d(t;q)=x(1-q)\sum_{k=0}^{\infty}q^k f(xq^k),$$

$$\int_x^{\infty} f(t)\,d(t;q)=x(1-q)\sum_{k=1}^{\infty}q^{-k} f(xq^{-k})$$

and

$$\int_0^{\infty} f(t)\,d(t;q)=(1-q)\sum_{k=-\infty}^{\infty}q^k f(q^k).$$

The fractional q-derivative operator of arbitrary order α, for a function $f(x)$ due to Al-Salam (1966) and Agarwal (1969) is defined as:

$$D_{x,q}^{\alpha}\{f(x)\}=\frac{1}{\Gamma_q(-\alpha)}\int_0^x(x-tq)_{-\alpha-1}f(t)\ d(t;q),$$

where $\mathrm{Re}(\alpha)<0.$

In particular, for $f(x) = x^{\mu-1}$

$$D_{x,q}^{\alpha}(x^{\mu-1}) = \frac{\Gamma_q(\mu)x^{\mu-\alpha-1}}{\Gamma_q(\mu-\alpha)},$$

where $\mu \neq \alpha\ ; \mu \neq 0, -1, -2, \ldots$.

Agarwal (1969) defined the q-analogue of the Riemann-Liouville fractional integral operator for a function $f(x)$ in the following manner:

$$I_q^{\mu}\ f(x) = \frac{1}{\Gamma_q(\mu)} \int_0^x (x - tq)_{\mu-1} f(t)\ d(t;q),$$

where μ is an arbitrary order of integration, such that $\mathrm{Re}(\mu) > 0.$

Al-Salam (1966) introduced the basic (or *q*-) analogue of the Weyl fractional integral operator of arbitrary order μ, in the following form:

$$K_q^{\mu}\ f(x) = \frac{q^{-\mu(\mu-1)/2}}{\Gamma_q(\mu)} \int_x^{\infty} (t - x)_{\mu-1} f(t\, q^{1-\mu})\ d(t;q),$$

where $\mathrm{Re}(\mu) > 0$ and $K_q^0\ f(x) = f(x)$.

Al-Salam (1966) and Agarwal (1969) introduced and studied in detail the theory and various properties of the fractional q-operators and Recently, Yadav and Purohit (2004a, 2005a and 2006a) obtained fractional q-integrals of Riemann-Liouville and Weyl type, for the generalized basic hypergeometric functions including the q-analogue of Fox's *H*-function given by equation .

1.8 The q-Laplace Transforms:

In the year 1949, Hahn (1949b) defined the basic (or *q*-) analogues of the well-known classical Laplace transforms

$$\varphi(s) = \int_0^{\infty} e^{-sx} f(x)\, dx\ ,\ s > 0$$

by means of the q-integral equations:

$${}_q L_s\, f(x) = \frac{1}{(1-q)} \int_0^{s^{-1}} E_q(qsx)\, f(x) d(x;q)$$

and

$$ {}_q\pounds_S\ f(x) = \frac{1}{(1-q)} \int_0^{\infty} e_q(-sx)\, f(x) d(x;q) \ , $$

where $\mathrm{Re}(s) > 0.$

Hahn also obtained the q-analogue of certain fundamental properties of the Laplace transform.

1.9 Saigo's fractional q-integral operators: The definition of Saigo's fractional q-integral operators has recently been provided by Purohit and Yadav (2010), with the restriction that one of the parameters, ε, must be a non-negative integer. Under that limitation, it was impossible to provide a definition of fractional derivatives. Garg and Chanchlani (2011) provide the definitions of the Saigo's fractional q-integral operator as follows to get over these issues. For $\vartheta \in \mathbb{C}, \Re(\vartheta) > 0, \zeta$ and ε being real or complex, the generalized fractional q-integral operators $\mathcal{I}_q^{\vartheta,\zeta,\varepsilon}(.) and\ \mathcal{g}_q^{\vartheta,\zeta,\varepsilon}(.)$defined in the following manner:

$$ \mathcal{I}_q^{\vartheta,\zeta,\varepsilon} f(z) = \frac{z^{-\zeta-1}}{\Gamma_q(\vartheta)} \int_0^z (tq/z;q)_{\vartheta-1} $$
$$ \times \sum_{n=0}^{\infty} \frac{\left(q^{\vartheta+\zeta};q\right)_n (q^{-\varepsilon};q)_n}{(q^{\vartheta};q)_n (q;q)_n} (-1)^n q^{(\varepsilon-\zeta)n} q^{-\binom{n}{2}} \left(\frac{t}{z} - 1\right)_q^n f(t) d_q t $$

and

$$ \mathcal{g}_q^{\vartheta,\zeta,\varepsilon} f(z) = \frac{q^{-\vartheta(\vartheta+1)/2-\zeta}}{\Gamma_q(\vartheta)} \int_z^{\infty} (z/t;q)_{\vartheta-1} t^{-\zeta-1} \times $$
$$ \sum_{n=0}^{\infty} \frac{\left(q^{\vartheta+\zeta};q\right)_n (q^{-\varepsilon};q)_n}{(q^{\vartheta};q)_n (q;q)_n} (-1)^n q^{(\varepsilon-\zeta)n} q^{-\binom{n}{2}} \left(\frac{z}{qt} - 1\right)_q^n f\left(tq^{1-\vartheta}\right) d_q t $$

where$0 < |q| < 1$, a real valued function $f(z)$ on $(0, \infty)$.

2. Literature Review

2.1 Introduction

A concise overview of the literature review conducted to finalize the current research proposal is presented in this section. The review process involved analyzing information from 60 research papers.

2.2 Review Process

The literature review can be divided into four stages as follows.

1. Initially, we conducted a search for research papers pertaining to our field of study from credible sources. This allowed us to pinpoint and explore our area of interest,

2. Our effort was directed towards categorizing the relevant research papers in our field of interest.

3. Our comprehensive analysis of the classified research papers has provided us with a thorough comprehension of the techniques employed to support the problem, the reasoning behind the importance and originality of the proposed approach, the specific inquiries addressed, the primary contribution made, as well as the extent and constraints of the presented work.

4. This phase involves synthesising the research papers' information, ideas, and findings.

This review process helped us to structure our line of study towards accomplishment of the research work.

2.3 Summary of the Literature Reviewed

A large number of research papers relevant to the subject of the study were examined during the evaluation process. To summarise, some researchers and their contributions to the literature are listed below, categorically:

$$J_{\eta}^{(2)}(z;q) = \frac{(q^{n+1};q)_{\infty}}{(q;q)_{\infty}} \sum_{r=0}^{\infty} \frac{q^{r(r+\eta)}(z/2)^{\eta+2r}}{(q,q^{\eta+1};q)_r},$$

$$J_{\eta}^{(3)}(z;q) = \frac{(q^{n+1};q)_{\infty}}{(q;q)_{\infty}} \sum_{r=0}^{\infty} \frac{q^{\left(\frac{r+1}{2}\right)}}{(q,q^{\eta+1};q)_r} (z/2)^{\eta+2r},$$

It obvious that

$$J_{\eta}^{(j)}(z;q) = e^{\frac{-i\pi\eta}{2}} J_{\eta}^{(j)}\left(e^{\frac{i\pi}{2}}z;q\right), \quad j \in \{1,2,3\}$$

Mansour and Al-Shomrani (2013) introduced the following q-Bessel function:

$$J_{\eta}^{(4)}(z;q) = \frac{(q^{n+1};q)_{\infty}}{(q;q)_{\infty}} (z/2)^{\eta}{}_0\emptyset_1 \left[\begin{matrix} -; \\ \\ q^{n+1}; \end{matrix} \quad q; \quad \frac{-q^{\frac{3(\eta+1)}{2}}z^2}{4}\right],$$

Which is a q-analogy of the modified Bessel function.

Mahmoud (2013) has published the following definition of the generalized q-Bessel function for $\xi \in Z, |z| < 2$ and $\xi = 0$ as follows.

$$J_{\eta}(z,\xi;q) = \frac{(z/2)^{\eta}}{(q;q)_{\eta}} \sum_{r=0}^{\infty} \frac{(-1)^{r(\xi+1)} q^{\left(\frac{\xi r(r+\eta)}{2}\right)}}{(q^{\eta+1};q)_r (q;q)_r} (z^2/4)^r$$

where

$$(q;q)_r = \prod_{i=1}^{r-1} \left(1 - q^{i+1}\right)$$

Reduced to q-Bessel functions of the first, second and third kind respectively if we substitute $\xi = 0; 2; 1$. $J_{\eta}(z,\xi;q)$ is a q-Bessel function $J_{\eta}(z)$ and modified Bessel function $J_{\eta}(z)$.

$$Lim_{q \to 1} J_{\eta}((1-q)z,\xi;q) = J_{\eta}(z), \qquad \xi = 0,2,4,\ldots$$

$$Lim_{q \to 1} J_{\eta}((1-q)z,\xi;q) = I_{\eta}(z), \qquad \xi = 1,3,5,\ldots$$

2.4 Reviewed based on q-analogue of Genaralized Hypergeometric Function

Vyas et al.(2021). The article explores four theorems related to the fractional integral image, specifically focusing on the product of the q-analogue of a general class of polynomials and the q-analogue of I-functions. The study utilizes q-fractional integrals of Erdelyi–Kober type and generalized Weyl type fractional operators. The findings highlight various results derived from the interplay between Erdelyi–Kober type and Riemann–Liouville q-fractional integrals, as well as the relationship between generalized Weyl type and Weyl type q-fractional integrals.

Vyas et al.(2020). The paper investigates the basic analogue of Laplace transforms, specifically focusing on the evaluation of the product of a general class of q-polynomials, along with the q-analogue of Fox's H-function and q-analogue of I-function. The study explores the limiting cases of these evaluations and demonstrates a diverse range of outcomes that can be obtained.

Vyas et al.(2019). The primary focus of this article is to ascertain the q-Sumudu transforms of a product involving a unified family of q-polynomials, along with the basic (or q-) analog of Fox's H-function and q-analog of I-functions. The study explores specialized cases, including the q-Sumudu transform of a general class of q-polynomials and the q-Sumudu transforms of the basic analogs of Fox's H-function and I-functions.

Noor and Razzaque(2022). The q-analogus of the differential operator connected to the q-Mittag-Leffler function was developed by the authors. A novel subclass of ana- lytical functions in conic domains, k-US" (a.), was developed by the authors using this newly described operator. Additionally, a variety of relevant qualities were looked into, including the structural formula, coefficient estimates, Fekete-Szego problem, and subordination outcome.

Sharma and Jain(2016). The authors explored the q-difference relation of the q- analogue of the generalized Mittag Leffler function using the q-calculus technique, as well as other features applying the 4-derivative.

Mehrez(2018). The goal of this study was to derive a number of Turan-type inequalitiesfor the q-Mittag-Leffler and q-Wright functions. Furthermore, the authors demonstrated the monotonicity of ratios for sections of series of q-Mittag Leffler and 4-Wright func tions, and the results were closely related to Turan-type inequalities. To reach some of the important results, researchers used the methods discovered for traditional Mittag- Leffler and Wright functions. They provided two open problems that may be of interest for future investigation

Bairwa et al.(2022b). The paper introduced and analysed some properties and ap- plications of the q-analogue of the generalized Mittag-Leffler function ${}_{\gamma,\delta}E_{\alpha,\beta}$. The researchers established some of its features, including the convergence condition, the Caputo fractional q-derivative, the Hilfer fractional q-derivative, the Kober fractional q-integral, the Jackson q-integral, and the q-Sumudu transform. Furthermore, they examined the applications of the q-analogue of the generalized Mittag-Leffler function in a kinetic equation and found that the solution to the fractional q-kinetics equation contains this function and explains certain particular instances.

Rehman et al.(2019). Authors aimed giving lower bounds for the ratio of some normalised q-Mittag-Leffler functions and their sequences of partial sums were the main goal of investigation. Researchers take into account a number of corollaries and impacts of your main conclusions.

Nadeem et al.(2020b). The Mittag-Leffler function was generalized in a novel way using -calculus. After that, the authors looked into its outstanding characteristics, in- cluding its recurrence relation, integral representation, q-derivative formula, q-Laplace transformation, and image formula under the q-derivative operator. Also, they take into account a few particular situations to illustrate how to use your key results.

Hadi et al.(2022). This article introduced and explored a newly created generalized q-Mittag-Leffler function. Using the idea of differential subordination and the newly defined function as inspiration, a new subclass of multivalent functions is introduced. They are given some geometric features. A generalized Srivastava-Aniya integral operator's related radii for the aforementioned we also investigated.

Yue et al.(2021). First, k-fractional integral operators incorporating further extensions of the Mittag-Leffler function are defined in this study. The first and second Hadamard and Fejer-Hadamard inequalities for generalized k-fractional integrals are then obtained. Finally, solutions for p-convex functions are achieved using these generalized k-fractional integrals, including Mittag-Leffler functions. Taking p = 1 yields the findings for convex functions.

Saenko(2022). The gamma function's integral representation is generalized in this study, and demonstrated how the Hankel contour allows rotation on the complex plane. The rotation angle of the contour has a set range of acceptable values. The writers established a generalisation of the integral form of the Mittag-Leffler function using this integral representation, which expresses the value of this function using the contour integral.

Ghanim et al.(2021). The study establishes new connections between the Mittag- Leffler functions of one to four parameters using Riemann-Liouville fractional calculus. The original Mittag-Leffler function is becoming more and more well-known, and more in-depth research is being done on it. As a result, it is likely that those findings will be extended in the future to simplify some significant physical models that use four or more Mittag-Leffler functions or to provide more useful mathematical models for these functions.

Khan and Khan(2020). The Mittag-Leffler function, developed by Shukla and Pra- japati, is expanded in this study using the extended beta function. The expanded Mittag- Leffler function's important characteristics and formulas, such as integral representation and Mellin transform, were investigated.

3. Problem Statement and Objectives

3.1 Problem Statement

The Mittag-Leffler function is a well-known mathematical concept that has found applications in various fields of science and engineering, including fractional calculus, probability theory, and complex analysis. This function generalizes the exponential function and is defined as a power series. In recent years, there has been growing interest in exploring q-analogues of various mathematical functions, which are modifications of these functions that involve an extra parameter q. The q-analogue of a function aims to extend its properties and applicability to new areas of research.

The problem statement at hand pertains to the investigation and analysis of the q analogue of the Mittag Leffler function. Specifically, the research aims to address the following key aspects

Definition and Formulation: Develop a comprehensive definition of the q-analogue of the Mittag-Leffler function and formulate its representation as a series or integral involving the parameter q Investigate how this definition relates to the classical Mittag- Leffler function when q approaches 1.

Properties and Special Cases: Explore the properties of the q-analogue of theMittag Leffler function, including its behavior for different values of q. Analyze the limiting cases as q approaches 1 and other significant values of q, and compare these properties with the classical Mittag-Leffler function. Investigate the convergence and divergence properties of the y-analogue series.

Fractional Calculus and Differential Equations: Examine the applications of the q-analogue of the Mittag-Leffler function in the context of fractional calculus Investigate how it can be used to model and solve q-differential equations involving fractional derivatives and integrals.

Probabilistic and Statistical Applications: Explore potential applications of the q analogue of the Mittag-Leffler function in probability theory, stochastic processes, and statistical

distributions. Investigate whether this q-analogue cum provide insights into phenomena that are not adequately captured by the classical Mittag-Leffler function.

Connection to Other q-Analogues: Establish connection between the y-analogue of the Mittag-Leffler function and other g-analogues of mathematical functions. Investigate whether any q-analogue identities or transformations involving the Mittag-Leffler function hold true for its analogue as well.

The successful exploration of the q-analogue of the Mittag-Leffler function and its properties has the potential to extend the applicability of this mathematical concept to new fields and problems. It may lead to novel insights, analytical tools, and computational techniques that contribute to the advancement of various scientific and engineering disciplines.

A study of the literature over the past twenty years showed that there has been very little research on generalized q-fractional operators with q-Mittag- Leffler functions to investigate different q-fractional calculus operators that appear in many disciplines of applied sciences and engineering

Based on the observations presented above, the planned study has been given the following title:

"Mathematical Analysis of Mittag-Leffler Functions in Quantum Calculus"

3.2 Objectives

- To develop new formulas for basic and advanced q-Mittag-Leffler functions using specified q-fractional integral and derivative operators of one or more variables.
- To examine the extension of the q-analogue of Mittag-Leffler functions and their important properties such as convergence, recurrence relation, integral representation, q-derivative formula, q-transform, and image formula under the q-derivative operator.
- To create q-integral transformations, representations, and formulas that involves q-Mittag-Leffler functions, operators, and properties.
- To investigate the possibility of generalizing different q-fractional calculus operators and determine their convergence conditions.

4. Gaps in Research Papers

The q-fractional theory of the generalized q-Hypergeometric function lacks the well-established status enjoyed by the fractional theory of the fundamental hypergeometric function. Nonetheless, our study aims to lay the groundwork for a more comprehensive q-fractional theory of the hypergeometric function.

- Current methods for fractional q-calculus and their applications across diverse scientific and technical domains are insufficient
- There is a need to extend crucial findings related to basic fractional calculus operators, integral transformations, and polynomials to their q-analogues. New results in q-calculus operators for the product of generalized hypergeometric functions, involving the q-analogue of a general class of polynomials and q-Mittag-Leffler functions, are eagerly anticipated.
- Research is ongoing to explore q-extensions of special functions like the I-function and H-functions. Defining characteristics such as convergence, recurrence relations, integral representations, q-derivative formulas, q-Laplace transformations, q-sumudu transforms, q-Saigo integral operators, and image formulas under the q-derivative operator are areas of interest.
- The application of fractional q-calculus operators, specifically those involving the q-hypergeometric function, to the theory of q-difference and q-integral equations, q-integral inequalities, and the theory of analytic functions remains an open avenue.
- The frequent usage of q-fractional calculus in the geometric theory of functions is under evaluation due to the absence of a defined complex domain. Geometric features for various applications, including coefficient estimates (bounds), radii of close convexity, starlikeness, convexity, and distortion qualities, are subjects of ongoing study.
- Ongoing research is directed towards exploring numerous integral inequalities using fractional q-calculus operators.

5.Research Methodology

In present research an attempt has been made to evaluate the q-Laplace transforms, q-sumudu transforms and other various type of integral transform of some basic hypergeometric functions of one, two and more variable by using some known results derived by various mathematicians.

6. Data Analysis

Method(s) of data collection: The data for the purpose of this study should be gathered qualitatively. As a result, data can be gathered from written documents and cross-referenced with established facts from research papers and standard books.

Method(s) of data analysis: The study's findings will be analyzed qualitatively through document analysis to verify them with established facts. As part of the researchers' investigations, they discovered q-fractional integrals of q-Mittag-Leffler functions using various q-analogues such as Riemann-Liouville fractional integrals, Weyl fractional integrals, Kober fractional integrals, Hadamard fractional derivatives.

7. Limitation

The strengths and limitations noted in the literature review are covered in this section.

7.1Strengths

- The application of basic (or q-) Mittag-Leffler functions has known benefits in a variety of fields, including number theory, combinatorial analysis, finite vector spaces, Lie theory, particle physics, nonlinear electric circuit theory, mechanical engineering, heat conduction theory, quantum mechanics, cosmology, and statistics.
- Using the q-differential operator and keeping recent developments in quantum calculus and its numerous applications in physics and applied mathematics in mind, the researchers proposed a novel class of Mittag-Leffler function, a subclass of univalent function, and a subclass of meromorphic multivalent function of these function domains. Furthermore, for the functions in these classes, the integral mean inequalities, close-to-convexity, inclusion property, extreme points, and distortion boundaries were identified.
- It is believed that the q-calculus and fractional q-calculus will inspire and p important future research. Along with a list of different characteristics and image formulas for fundamental Mittag-Leffler functions, a formal form of the q-fractional Riemann-Liouville operators and their generalized q-fractional operators was also provided.
- The Laplace transform method has traditionally been used to resolve to value problems and differential equations. The generalized kinetic equations, fractional time heat

equations, and fractional space-time diffusion equations are solved using various strategies.

7.2 Weaknesses

- For q-calculus to move forward, a notation must be used correctly.
- The image formulae may be useful in solving fundamental (q-) differential or (q-) integral equations. Because there is no systematic and unified development of different image formulas under generalized q-fractional Mittag-Leffler function, dealing with q-calculus has several limitations.
- When the quantum parameter q goes to one, the introduction of the q-analogue of fractional calculus operators in q-calculus is extremely difficult because there may be numerous of them and they all must unquestionably satisfy the limiting formula.
- Fractional calculus operators are significantly used in the geometric theory of Mittag-Leffler functions. However, because there is no definition for complex domains, the use of q-fractional calculus is currently being investigated.
- Currently, the q-calculus theory is not used to address practical engineering problems.

7. Work Plan Of The Research

The dissertation would be finished between March 2024 and May 2025. The planned Ph.D. dissertation project will need the following time commitment:

S.No.	Proposed work	Time duration
1	To develop new formulas for basic and advanced q-MittagLeffler functions using specified q-fractional operators of one or more variables.	4 month
2	To examine the extension of the q-analogue of MittagLeffler functions and their important properties such as convergence, recurrence relation, integral representation, q- derivative formula, q-transform, and image formula under the q-derivative operator.	2 month
3	To create q-integral transformations, representations, and formulas that involve 4-Mittag-Leffler functions, operators, and properties.	3 month

4	To investigate the possibility of generalizing different fractional calculus operators and determine their convergence conditions.	3 month
5	To explore how q-calculus operators involving the q-Mittag- Leffler function can be applied in scientific and engineering Scenarios.	3 month

8. References

1. Abdeljawad, T., Alzabuit, J., and Baleanu, D. (2016). A generalized q-fractional gronwall inequality and its applications to nonlinear delay q-fractional difference systems. Journal of Inequalities and Applications, 2016(1):1-13.

2. Agarwal, H. (1989). On certain basic hypergeometric functions of several variables. Indian J. Math, 31(1):81-87.

3. Agarwal, R. P. (1969). Certain fractional q-integrals and q-derivatives. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 66, pages 365-370 Cambridge University Press.

4. Al-Salum, W. A. (1966). Some fractional q-integrals and q-derivatives. Proceedings of the Edinburgh Mathematical Society, 15(2):135-140.

5. Albayrak, D., Purohit, S. D., and Ucar, E. (2012). On q-integral transforms and their applications. Bulletin of Mathematical Analysis and Applications, 4:103-115.

6. Albayrak, D., Purohit, S. D., and Uçar, F. (2013). On q-analogues of sumudu transform. Analele Universitatii Ovidius" Constanta-Seria Matematica, 21(1):239-259.

7. Annaby, M. H. and Mansour, Z. S. (2012). q-Fractional calculus and equations, volume 2056. Springer.

8. Asley, R. (1978). The q-gamma and q-beta functions. Applicable analysis, 8(2):125-141.

9. Bairwa, R., Kumur, A., and Kumur, D. (2022a). Certain properties of generalized q-mittag-leffler type function and its application in fractional q-kinetic equation. International Journal of Applied and Computational Mathematics, 8(5):1-13.

10. Bairwa, R., Kumar, A., and Kumb). Certain properties of generalized q mittag-leffler type function and its tion in fractional q-kinetic equation. International Journal of Applied and Computational Mathematics, 8(5):219.

11. Bairwa, R., Kamar, A., and Kumar, D. (2026). Certain properties of generalized q mittag-leffler type function and its application in fractional q-kinetic equation. International Journal of Applied and Computational Mathematics, 8(5):219.
12. Balayan, D., Purohit, S. D., and Uçar, F. (2013). On q-analogues of sumudu transform. Analele Universitatii Ovidius" Constanta-Seria Matematica, 21(1):239-259.
13. Gabara, H. M. and Shokerilla, A. A. (2021). Mellin transform of mittag-leffler density and its relationship with some special functions. International Journal of Research in Engineering and Science (LIRES), 9(3):08-13.
14. Garg, M. and Chanchlani, L. (2011). q-analogues of saigo's fractional calculus operators. Bull. Math. Anal. Appl, 3(4):169-179.
15. Gasper, G. (1990). M. rahman-basic hypergeometric series. Encyclopedia of Mathematics and its Applications, 35.
16. Gasper, G. and Rahman, M. (2004). Basic lepergeometric series, volume 96. Cambridge university press.
17. Ghanim, F. Al-Janaby, H. F., and Bazighifan, O. (2021). Some new extensions on fractional differential and integral properties for mittag-leffier confluent hypergeometric function. Fructal and Fractional, 5(4):143.
18. Hadi, S. H., Darus, M., Lee, J. R., et al. (2022). Some geometric properties of multivalent functions associated with a new generalized q-mittag-leffler function. AIMS Mathematics, 7(7):11772-11783.
19. Ismail, M. and Zhang, R. (2018). q-bessel functions and rogers-ramanujan type identities. Proceedings of the American Mathematical Society, 146(9):3633-3646.
20. Ismail, M. and Zhang, R. (2018). nctions and rogers-ramanujan type identities. Proceedings of the American Mathematical Society, 1469):3633-3646.
21. Jackson, F. (1905a). The application of basic numbers to bessel's and legendre's functions. Proceedings of the London Mathematical Society, 2(1):192–220.
22. Khan, N. and Khan, S. (2020). A new extension of the mittag-leffler function. Palestine Journal of Mathematics, 9(2):977-983.
23. Khan, S. (2021). A study of unified integrals involving generalized mittag-leffler function (gmlf). Acta Universitatis Apulensis Mathematics-Informatics, (67).
24. Mahmoud, M. (2013). Generalized q-bessel function and its properties. Advances in Difference Equations, 2013(1):1-11.
25. Mansour, M. and Al-Shomrani, M. (2013). New q-analogue of modified bessel function and the quantum algebra (2). J. Comput. Anal. Appl. 15:655-664.

26. Mansour, Z. S. (2009). Linear sequential q-difference equations of fractional order. Fract. Calc. Appl. Anal, 12(2):159-178.
27. Meena Kumari Gurjar, P. C. and Shrivastava, S. C. (2021). Fractional kinetic equations involving the mittag-leiffer (p; s; k) function via sumudu transform. Jnanabha, 51:2.
28. Mehrez, K. (2018). Turan type inequalities for q-mittag-leffler and q-wright functions. Math Ineq. Appl. 21:1135-1151.
298. Mittag-Leffler, G. (1903), Sur la novelle function.,(x). C.R. Acad. Sci. Paris, 137:554-558.
30. Nadeem, R., Saif, M., Usman, T., and Khan, A. H. (2019). New approach to generalized mittag-leffler function via quantum calculus. arXiv preprint arXiv: 1901.05851.
31. Nadeem, R.. Usman. T., Nisar, K. S. and Abdeljawad. T. (2020). A new generalization of mittag-leffler function via q-calculus. Advances in Difference Equations, 2020(1):1-10.
32. Nadeem. R.. Usman, T., Nisar, K. S., and Abdeljawad, T. (2020b). A new generalization of mittag-leffler function via q-calculus. Advances in Difference Equations, 2020:1-10.
33. Nisar, K. S. (2020), Generalized mittag-leffler type function: fractional integrations and application to fractional kinetic equations. Frontiers in Physics, 8:33.
34. Nisar, K. S. Suthar, D., Agarwal, R., and Purohit, S. (2020). Fractional calculus operators with appell function kerr mittag-leffler function. Advanced to srivastava polynomials and extended ce Equations, 2020 (1):1-14.
35. Prabhakar, T. R. et al. (1971). A singular integral equation with a generalized mittag-letter function in the kernel. Yokohama math J, 19(1):7-15.
36. Purohit, S. D. and Kalla, S. L. (2011). A generalization of q-mittag-leffler function. Mat, Bilten, 35:15-26.
37. Purohit, S. D. and Ucar, F. (2018). An application of q-sumudu transform for fractional q-kinetic equation. Turkish Journal of Mathematics, 42(2):726-734.
38. Purohit, S. D. and Yadav, R. K. (2010). On generalized fractional q-integral operators involving the q-gauss hypergeometric function. Bull. Math. Anal. Appl, 2(4):35-44.
39. Rajković, P. M., Marinković, S. D., and Stanković, M. S. (2007). Fractional integrals and derivatives in q-calculus Applicable analysis and discrete mathematics, pages 311-323.
40. Rehman, M., Ahmad, Q. Z. Srivastava, H. M. Khan, B., and Khan, N. (2019). Partial sums of generalized q-minag-leffler functions. Aims Math, 5(1):408-420.
41. Salem, A. and Ucar, F. (2016). Fractional calculus on n-d mittag-leffler function and its properties. Turkish Journal of Mathematics, 40(4):961-972.

42. Salim, T. (2009). Some properties relating to the generalized mittag-leffler function. Advances in Applied Mathematical Analysis, 4-21-30.
43. Salim, T. and Faraj, A. (2012) A generalization of mittag leffler function and integral operator associated with fractional calculus. Journal of Fractional Calculus and Applications, 3(5):1-13.
44. Saxena, R. and Kumar, R. (1995). A basic analogue of the generalized h-function. Le Matematiche, 50(2):263–271.
45. Saxena, R., Modi, G., and Kalla, S. (1983). A basic analogue of fox's h-function. Rev Tec. Ing. Univ. Zulia, 6(1):139–143.
46. Sharma, S. and Jain, R. (2014). On some properties of generalized q-mittag leffler function. Mathematica Aeterna, 4(6):613-619.
47. Sharma, S. and Jain, R. (2016). On some recurrence relations of generalized q-mittag leffler function. Math. Aeterna, 6:791-795.
48. Vyas, V. K., Al–Jarrah, A. A., & Purohit, S. D. (2019). q-Sumudu transforms pertaining to the product of family of q-polynomials and generalized basic hypergeometric functions. Applications and Applied Mathematics: An International Journal (AAM), 14(2), 30.
49. Vyas, V. K., Al-Jarrah, A. A., Suthar, D. L., & Abeye, N. (2021). Fractional q-Integral Operators for the Product of a q-Polynomial and q-Analogue of the I-Functions and Their Applications. Mathematical Problems in Engineering, 2021, 1-9.
50. Yang, W. (2022). Certain new chebyshev and grüss-type inequalities for unified fractional integral operators via an extended generalized mittag-leffler function. Fractal and Fractional, 6(4):182.
51. Yadav, R. K. and Purohit, S. D. (2004). Application of riemann-liouville fractional q-integral operator to basic hypergeometric functions. Acta Ciencia Indica, 30(3):593--600.
52. Yadav, R. K. and Purohit, S. D. (2006). On applications of weyl fractional q-integral operator to generalized basic hypergeometric functions. Kyungpook Mathematical Journal, 46(2):235-245.
53. Yue, Y., Farid, G., Demirel, A. K., Nazeer, W., and Zhao, Y. (2021). Hadamard and fejér-hadamard inequalities for generalized k-fractional integrals involving further extension of mittag-leffler function. AIMS Mathematics, 7(1):681-703.
54. Zhang, Z., Farid, G., Mehmood, S., Jung, C.-Y., and Yan, T. (2022). Generalized k-fractional chebyshev-type inequalities via mittag-leffler functions. Axioms, 11(2):82.